DEFORMATIONS OF PRODUCTS AND NEW EXAMPLES OF OBSTRUCTED IRREGULAR SURFACES

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Abstract. We determine the base space of the Kuranishi family of some complete intersection in the product of an abelian variety and a projective space. As a consequence we obtain new examples of obstructed irregular surfaces with ample canonical bundle and maximal Albanese dimension.

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Introduction

Define an equivalence relation on analytic singularities generated by: *if* $(X,p)\to(Y,q)$ *is a smooth morphism, then* $(X,p)\sim(Y,q)$. We call the equivalence classes *singularity types*. It is easy to prove that every analytic singularity is determined, up to isomorphism, by its singularity type and the dimension of its Zariski tangent space.

For every complex manifold X we denote by $\operatorname{Def}(X)$ the deformation space of X, i.e. the base space of its semiuniversal deformation. Thus $\operatorname{Def}(X)$ is determined by its singularity type and by the dimension of the cohomology group $H^1(X,T_X)$.

In the paper [35], Ravi Vakil shows that the moduli space of regular surfaces satisfies the "Murphy's law". More precisely he proves that every singularity type defined over $\mathbb{Z}$ is obtained as deformation space of a regular surface. The methods used there fail when applied to varieties X with $H^1(\mathcal{O}_X)\neq 0$; on the other side there exist in literature several examples of obstructed surfaces either admitting irrational pencils ([19], [23]) or containing nodal curves ([3], [4], [20]).

The aim of this paper is to extend some of the technical tools used in [35] to irregular surfaces; as a by-product we obtain some new easy examples of obstructed irregular surfaces.

Example. Let A be an abelian surface and S a smooth surface of general type contained in $A\times\mathbb{P}^1$. Then $\operatorname{Def}(S)$ has the same singularity type of the affine cone over the Segre variety $\mathbb{P}^2\times\mathbb{P}^1\subset\mathbb{P}^5$ (Example 7.5).

The same ideas can be used to understand the deformation type of certain complete intersections in $A\times\mathbb{P}^{n_1}\times\cdots\times\mathbb{P}^{n_k}$. For example we prove (Theorem 7.6):

Theorem A. *Let A be an abelian variety of dimension q, let D be a sufficiently ample divisor on $A\times\mathbb{P}^{n-1}$ and $S\subset A\times\mathbb{P}^{n-1}$ the intersection of m generic hypersurfaces homologically equivalent to D.*

If $0\le m\le q+n-3$, then $\operatorname{Def}(S)$ has the same singularity type of the commuting variety

$$C(q,\mathfrak{sl}(n,\mathbb{C}))=\{(A_1,\dots,A_q)\in\mathfrak{sl}(n,\mathbb{C})^{\oplus q}\mid A_iA_j=A_jA_i \text{ for every } i,j\}.$$

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Since two matrices in $\mathfrak{sl}(2,\mathbb{C})$ commute if and only if they are linearly dependent, the commuting variety $C(q,\mathfrak{sl}(2,\mathbb{C}))$ is isomorphic to the affine cone over the Segre variety $\mathbb{P}^{q-1}\times\mathbb{P}^2\subset\mathbb{P}^{3q-1}$. By a classical result ([11], [30]) the commuting variety $C(2,\mathfrak{sl}(n,\mathbb{C}))$ is irreducible for every n, while $C(q,\mathfrak{sl}(n,\mathbb{C}))$ is reducible for every pair of positive integers (q,n) such that the product $(q-3)(n-3)$ is sufficiently large (Lemma 3.1).

The proof of Theorem A is divided in two parts. In the first we study deformations of products of Kähler manifolds (hence the name of this paper) using the theory of differential graded Lie algebras. The main results is the Formality Theorem 4.3 which implies in particular that, under suitable cohomological condition on X,Y, the deformation space of $X\times Y$ is completely determined by the primary obstruction map of Kodaira and Spencer [22].
The second part is essentially a works about stability and costability theorems; more precisely we prove analogs of Horikawa's theorems [15, 16, 17] in some cases where Horikawa's hypothesis are not completely satisfied.

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1. General notation and basic facts

We work over the field $\mathbb{C}$ of complex numbers; every complex manifold is assumed compact and connected.
For every complex manifold X we denote by:

- $B_X=\oplus_i H^i(X,\mathcal{O}_X)$ the graded algebra of the cohomology of the structure sheaf of X, endowed with the cup product $\wedge$.
- For every holomorphic vector bundle E on X let $\mathcal{A}^{p,q}_X(E)$ be the sheaf of differentiable (p,q)-forms of X with values in E and $A^{p,q}_X(E)=\Gamma(X,\mathcal{A}^{p,q}_X(E))$ the space of its global sections.
- T_X the holomorphic tangent bundle of X.
- $K_X=A^{0,*}_X(T_X)=\oplus_i\Gamma(X,\mathcal{A}^{0,i}_X(T_X))$ the Kodaira-Spencer differential graded Lie algebra of X.
- $\operatorname{Def}(X)$ the deformation space of X, i.e. the base space of the semiuniversal deformation of X.
- **Art** the category of local artinian $\mathbb{C}$-algebras.
- If L is a differential graded Lie algebra (DGLA), we denote by
$$\operatorname{Def}_L=\frac{\text{Maurer-Cartan}}{\text{Gauge action}}\colon \mathbf{Art}\to\mathbf{Set}$$
the associated deformation functor (see [13], [24], [27] for precise definition and properties).

In this paper we shall need several times the following result (for a proof see e.g. Theorem 5.71 of [27]).

Theorem 1.1 (Schlessinger-Stasheff [34]). *Let $L\to M$ be a morphism of differential graded Lie algebras. Assume that:*

(1) $H^0(L)\to H^0(M)$ *is surjective.*
(2) $H^1(L)\to H^1(M)$ *is bijective.*
(3) $H^2(L)\to H^2(M)$ *is injective.*

Then $\mathrm{Def}_L \to \mathrm{Def}_M$ *is an isomorphism.*

If X, Y are complex manifolds and $E \to X$, $F \to X$ vector bundles, we denote $E \boxtimes F = p^*E \otimes q^*F$, where $p\colon X \times Y \to X$ and $q\colon X \times Y \to Y$ are the projections. By Künneth formula we have

$$H^i(X \times Y, E \boxtimes F) = \bigoplus_j H^j(X, E) \otimes H^{i-j}(Y, F).$$

2. Twisted deformations of complex manifolds

Let's recall the construction of the germ $\mathrm{Def}(X)$ for a compact complex manifold X. The starting point is the Kodaira-Spencer differential graded Lie algebra

$$K_X = A_X^{0,*}(T_X).$$

Fix a hermitian metric on X and denote by $d^*\colon A_X^{0,1}(T_X) \to A_X^{0,0}(T_X)$ the formal adjoint of d. Then we consider suitable Sobolev completion (we may use the appendix of [8] as instruction booklet)

$$A_X^{0,i}(T_X) \subset L^i$$

such that the induced maps of Hilbert spaces

$$d\colon L^1 \to L^2, \qquad d^*\colon L^1 \to L^0, \qquad [-,-]\colon L^1 \otimes L^1 \to L^2.$$

are defined and bounded.
Then (see e.g. [13]) $\mathrm{Def}(X)$ is isomorphic to the germ at 0 of the analytic subvariety of L^1 defined by the equations

$$dx + \frac{1}{2}[x, x] = 0, \qquad d^*x = 0.$$

According to elliptic regularity, Hodge theory on compact manifolds and the results of [13], the germ $\mathrm{Def}(X)$ is finite dimensional and well defined; in particular different choices of metric and Sobolev norms give isomorphic germs.

Remark 2.1. If the Kodaira-Spencer algebra K_X is formal, i.e. quasiisomorphic to its cohomology, then by [12], [13], the space $\mathrm{Def}(X)$ is analytically isomorphic to the germ at 0 of the nullcone of the quadratic map

$$H^1(X, T_X) \to H^2(X, T_X), \qquad \theta \mapsto [\theta, \theta].$$

In general the Kodaira-Spencer DGLA is not formal. In fact, it is a consequence of [9] that, if X is the Iwasawa manifold then $K_{X \times \mathbb{P}^1}$ is not formal; more generally Ravi Vakil proved [35], putting together the results of [29] and [25], that for every analytic singularity $(U, 0)$ defined over $\mathbb{Z}$ there exists a complex surface S with very ample canonical bundle such that $\mathrm{Def}(S) \cong (U \times \mathbb{C}^n, 0)$ for some integer $n \geq 0$.
Choosing $U = \{(x, y) \in \mathbb{C}^2 \mid xy(x - y) = 0\}$ and taking S as above, the DGLA K_S cannot be formal.

Assume now that $(B, \wedge)$ is a finite dimensional graded algebra of non negative degrees, say $B = \oplus_{i \geq 0} B^i$ and $\dim_{\mathbb{C}} B < +\infty$.
Taking the natural extensions of the operators d, d^* and $[\ ,\ ]$ on $L \otimes B$,

$$d_B(x \otimes h) = dx \otimes b, \quad d_B^*(x \otimes b) = d^*x \otimes b, \quad [x \otimes b, y \otimes c]_B = (-1)^{\overline{b}\,\overline{y}}[x, y] \otimes (b \wedge c),$$

we may define an analytic germ

$$\mathrm{Def}(X,B)=\{x\in(L^1\otimes B^0)\oplus(L^0\otimes B^1)\mid d_B^*x=0,\ d_Bx+\frac{1}{2}[x,x]_B=0\}.$$

According to [13, Thm's 3.9, 3.11] (see also [24, Thm. 4.7]), the germ $\mathrm{Def}(X,B)$ is a hull (in the sense of [33]) of the functor

$$\mathrm{Def}_{K_X\otimes B}\colon \mathbf{Art}\to\mathbf{Set}$$

and then its Zariski tangent space is isomorphic to $(H^1(T_X)\otimes B^0)\oplus(H^0(T_X)\otimes B^1)$, while its obstruction space is contained in

$$H^2(K_X\otimes B)=(H^2(T_X)\otimes B^0)\oplus(H^1(T_X)\otimes B^1)\oplus(H^0(T_X)\otimes B^2).$$

Note moreover that if K_X is formal, then also $K_X\otimes B$ is formal.

Assume now that B has a unit 1 and $\dim B^0=1$; if $b_1,\dots,b_q$ is a basis of B^1, then $\mathrm{Def}(X,B)$ is the set of pairs $(x,\sum_i y_i\otimes b_i)\in L^1\oplus(L^0\otimes B^1)$ such that

$$d^*x=0,\quad dx+\frac{1}{2}[x,x]=0,\quad \sum_i(dy_i+[x,y_i])\otimes b_i=0,\quad \sum_{i<j}[y_i,y_j]\otimes b_i\wedge b_j=0.$$

The inclusion $\mathbb{C}=B^0\subset B$ induces a closed embedding $\mathrm{Def}(X)\subset\mathrm{Def}(X,B)$, while the projection $B\to B^0$ induces an analytic retraction $r\colon \mathrm{Def}(X,B)\to\mathrm{Def}(X)$.
The fiber $r^{-1}(0)$ is the germ at 0 of

$$\left\{\sum_i y_i\otimes b_i\in L^0\otimes B^1\mid dy_i=0,\quad \sum_{i<j}[y_i,y_j]\otimes b_i\wedge b_j=0\right\}.$$

Since the kernel of $d\colon L^0\to L^1$ is $H^0(T_X)$, the above set is equal to

$$\{\sum_i y_i\otimes b_i\in H^0(T_X)\otimes B^1\mid \sum_{i<j}[y_i,y_j]\otimes b_i\wedge b_j=0\}.$$

In particular, if the product $\wedge\colon \bigwedge^2 B^1\to B^2$ is injective, then $r^{-1}(0)$ is isomorphic to the commuting variety

$$C(q,H^0(T_X))=\{(y_1,\dots,y_q)\in H^0(T_X)^{\oplus q}\mid [y_i,y_j]=0\text{ for every } i,j\}.$$

3. Basic facts about commuting varieties

Let L be a finite dimensional complex Lie algebra and q a positive integer.
The affine scheme

$$C(q,L)=\{(a_1,\dots,a_q)\in L^{\oplus q}\mid [a_i,a_j]=0\ \text{ for every } i,j\}.$$

is called the q-th commuting variety of L. Clearly $C(q,L\oplus M)=C(q,L)\times C(q,M)$; while if $p\le q$, then the projection on the first factors $C(q,L)\to C(p,L)$ is surjective; in particular if $C(q,L)$ is irreducible, then also $C(p,L)$ is irreducible.

The structure of the varieties $C(q,L)$ has been studied by several people. The case $L=\mathfrak{sl}(n,\mathbb{C})=H^0(T_{\mathbb{P}^{n-1}})$ has been studied in Gerstenhaber [11]; he proved in particular that $C(2,\mathfrak{sl}(n,\mathbb{C}))$ is irreducible for every n (this fact was also proved independently

by Motzkin and Taussky [30]). It is a well known open (and hard) problem to determine whether $C(2,\mathfrak{sl}(n,\mathbb{C}))$ (defined by the ideal generated by brackets) is a reduced scheme. Moreover, according to Richardson [32], the variety $C(2,L)$ is irreducible for every reductive Lie algebra L.

Lemma 3.1. *If the commuting variety $C(q,\mathfrak{sl}(n,\mathbb{C}))$ is irreducible then*

$$q < 3 + \frac{8n-12}{(n-2)^2}, \qquad \textit{for } n \textit{ even},$$

$$q < 3 + \frac{8}{n-3}, \qquad \textit{for } n \textit{ odd}.$$

Proof. This proof is based on the ideas of [18]. Assume $n \geq 4$, $C(q,\mathfrak{sl}(n))$ irreducible and consider the projection on the first factor

$$\pi\colon C(q,\mathfrak{sl}(n,\mathbb{C})) \to C(1,\mathfrak{sl}(n,\mathbb{C})) = \mathfrak{sl}(n,\mathbb{C}).$$

Let $D \in \mathfrak{sl}(n,\mathbb{C})$ be a diagonal matrix with distinct eigenvalues, then every matrix commuting with D must be diagonal. Therefore the fiber $\pi^{-1}(D)$ is irreducible of dimension $(n-1)(q-1)$ and the dimension of $C(q,\mathfrak{sl}(n,\mathbb{C}))$ is less than or equal to $n^2-1+(n-1)(q-1)$.
On the other hand, let r be the integral part of $n/2$ and let $N \subset \mathfrak{sl}(n,\mathbb{C})$ be the closed subset of matrices A such that $A^2=0$. It is easy to see that N is irreducible of dimension $2r(n-r)$ and therefore, for the generic $A \in N$, we have

$$\dim \pi^{-1}(A) < n^2 - 1 - 2r(n-r) + (n-1)(q-1).$$

After a possible change of basis, every $A \in N$ belongs to the space

$$H = \{(h_{ij}) \in \mathfrak{sl}(n,\mathbb{C}) \mid h_{ij} \neq 0 \text{ only if } i > r,\ j \leq r\}.$$

Then H is an abelian subalgebra of $\mathfrak{sl}(n,\mathbb{C})$ and therefore $\{A\} \times H^{\oplus q-1} \subset \pi^{-1}(A)$. In particular

$$r(n-r)(q-1) = \dim H^{\oplus q-1} < n^2 - 1 + (n-1)(q-1) - 2r(n-r).$$

A straightforward computation gives the inequalities of the lemma. □

Remark 3.2. If $\mathfrak{R}(\mathbb{Z}^q, \mathrm{PGL}(n,\mathbb{C}))$ is the space of representations $\rho\colon \mathbb{Z}^q \to \mathrm{PGL}(n,\mathbb{C})$, then the exponential gives an isomorphism between the germ at 0 of the commuting variety $C(q,\mathfrak{sl}(n,\mathbb{C}))$ and a neigbourhood of the trivial representation in $\mathfrak{R}(\mathbb{Z}^q, \mathrm{PGL}(n,\mathbb{C}))$.

4. Deformations of products

Let X, Y be two compact complex manifolds and denote by

$$\mathcal{X} \to \mathrm{Def}(X), \qquad \mathcal{Y} \to \mathrm{Def}(Y)$$

their semiuniversal deformations. The product $\mathcal{X} \times \mathcal{Y} \to \mathrm{Def}(X) \times \mathrm{Def}(Y)$ is a deformation of $X \times Y$ and therefore induces a morphism of analytic germs

$$\alpha\colon\ \mathrm{Def}(X) \times \mathrm{Def}(Y) \to \mathrm{Def}(X \times Y).$$

Lemma 4.1. *The morphism α is a closed embedding; it is an isomorphism if and only if $H^0(T_X) \otimes H^1(\mathcal{O}_Y) = H^1(\mathcal{O}_X) \otimes H^0(T_Y) = 0$.*

Proof. By Künneth formula

$$H^1(T_{X\times Y})\simeq H^1(T_X)\oplus H^1(T_Y)\oplus (H^0(T_X)\otimes H^1(\mathcal{O}_Y))\oplus (H^1(\mathcal{O}_X)\otimes H^0(T_Y)).$$

A morphism of analytic germs is a closed embedding if and only if it is injective on Zariski tangent spaces and the differential of α is equal to the natural embedding

$$H^1(T_X)\oplus H^1(T_Y)\to H^1(T_{X\times Y}).$$

The obstruction map associated to α is

$$H^2(T_X)\oplus H^2(T_Y)\to H^2(T_{X\times Y})$$

and, again by Künneth formula, it is injective. Therefore, if $H^0(T_X)\otimes H^1(\mathcal{O}_Y) = H^1(\mathcal{O}_X)\otimes H^0(T_Y)=0$, then the differential of α is bijective and α is an isomorphism. □

The condition $H^0(T_X)\otimes H^1(\mathcal{O}_Y)=H^1(\mathcal{O}_X)\otimes H^0(T_Y)=0$ is satisfied in most cases; for instance, a theorem of Matsumura [28] implies that $H^0(T_X)=0$ for every compact manifold of general type X.
If $H^1(\mathcal{O}_X)\otimes H^0(T_Y)\neq 0$, then it is easy to describe deformations of $X\times Y$ that are not a product. Assume for simplicity X Kähler, then $b_1(X)\neq 0$ and there exists at least one surjective homomorphism $\pi_1(X)\xrightarrow{g}\mathbb{Z}$. On the other hand, since $H^0(T_Y)\neq 0$, there exists at least a nontrivial one parameter subgroup $\{\theta_t\}\subset \operatorname{Aut}(Y)$, $t\in\mathbb{C}$, of holomorphic automorphisms of Y.
Therefore we get a family of representations

$$\rho_t\colon \pi_1(X)\to \operatorname{Aut}(Y),\qquad \rho_t(\gamma)=\theta_t^{g(\gamma)},\qquad t\in\mathbb{C}$$

inducing a family of locally trivial analytic Y-bundles over X.
Moreover, Kodaira and Spencer [22] proved that projective spaces $\mathbb{P}^n$ and complex tori $(\mathbb{C}^q/\Gamma)$ have unobstructed deformations, while the product $(\mathbb{C}^q/\Gamma)\times\mathbb{P}^n$ has obstructed deformations for every $q\geq 2$ and every $n\geq 1$ [22, page 436]. This was the first example of obstructed manifold; the same example is discussed, with a different approach, also in [9].

Lemma 4.2. *For every pair of compact Kähler manifolds X,Y there exists an injective quasi-isomorphism of differential graded Lie algebras*

$$(B_X\otimes K_Y)\oplus (B_Y\otimes K_X)\to K_{X\times Y}.$$

Proof. Assume X,Y compact complex manifolds and denote by

$$p\colon X\times Y\to X,\qquad q\colon X\times Y\to Y$$

the projections. Since $T_{X\times Y}$ is the direct sum of $p^*T_X=T_X\boxtimes\mathcal{O}_Y$ and $q^*T_Y=\mathcal{O}_X\boxtimes T_Y$ we have

$$H^i(X\times Y,T_{X\times Y})=H^i(X\times Y,p^*T_X)\oplus H^i(X\times Y,q^*T_Y)$$

and, by Künneth formula

$$H^i(X\times Y,p^*T_X)=\bigoplus_j H^j(T_X)\otimes H^{i-j}(\mathcal{O}_Y)=\bigoplus_j H^j(T_X)\otimes B_Y^{i-j},$$

$$H^i(X\times Y,q^*T_Y)=\bigoplus_j H^j(\mathcal{O}_X)\otimes H^{i-j}(T_Y)=\bigoplus_j B_X^j\otimes H^{i-j}(T_Y).$$

The isomorphism $T_{X\times Y}=p^*T_X\oplus q^*T_Y$ allows to define two injective morphisms of differential graded Lie algebras

$$p^*\colon K_X\to K_{X\times Y},\qquad q^*\colon K_Y\to K_{X\times Y}.$$

We note that p^* is injective in cohomology and the image of $H^i(T_X)$ is the subspace $H^i(T_X)\otimes H^0(\mathcal{O}_Y)\subset H^i(X\times Y,p^*T_X)$; similarly for the morphism q^*.
Denote by $\overline{\Omega}^*_X$ the graded algebra of antiholomorphic differential forms on X, endowed with the wedge product. More precisely $\overline{\Omega}^*_X=\oplus\overline{\Omega}^i_X$ where

$$\overline{\Omega}^i_X=\ker(\partial\colon \Gamma(X,\mathcal{A}^{0,i}_X)\to\Gamma(X,\mathcal{A}^{1,i}_X)).$$

If X is compact Kähler, then the natural maps

$$\overline{\Omega}^*_X\cap\ker\overline{\partial}\to\overline{\Omega}^*_X,\qquad \overline{\Omega}^*_X\cap\ker\overline{\partial}\to B_X=\frac{\ker\overline{\partial}}{\operatorname{Im}\overline{\partial}}$$

are both isomorphisms (see e.g. [7]) and therefore there exists an isomorphism of graded algebras $\overline{\Omega}^*_X\cong B_X$ independent from the Kähler metric. Denote also by $\overline{\Omega}^*_Y$ the graded algebra of antiholomorphic differential forms on Y.
We can define two morphisms

$$h\colon \overline{\Omega}^*_Y\otimes K_X\to K_{X\times Y},\qquad h(\phi\otimes\eta)=q^*(\phi)\wedge p^*(\eta).$$

$$k\colon \overline{\Omega}^*_X\otimes K_Y\to K_{X\times Y},\qquad k(\psi\otimes\mu)=p^*(\psi)\wedge q^*(\mu).$$

It is straightforward to check that h,k are morphisms of differential graded Lie algebras and that the image of h commutes with the image of k. This implies that the map

$$h\oplus k\colon (\overline{\Omega}^*_X\otimes K_Y)\oplus(\overline{\Omega}^*_Y\otimes K_X)\to K_{X\times Y}$$

is a morphism of differential graded Lie algebras. If X and Y are both Kähler manifolds, then according to Künneth formula the morphism $h\oplus k$ is a quasiisomorphism. □

Theorem 4.3. *Let X,Y be compact Kähler manifolds. Then there exists an isomorphism of germs*

$$\mathrm{Def}(X\times Y)\simeq\mathrm{Def}(X,B_Y)\times\mathrm{Def}(Y,B_X).$$

Moreover if K_X and K_Y are formal differential graded Lie algebras, then also $K_{X\times Y}$ is formal.

Proof. By Artin's theorem on solutions of analytic equations [1], it is sufficient to show that there exists a formal isomorphism.
The germ $\mathrm{Def}(X\times Y)$ is a hull of the functor $\mathrm{Def}_{K_{X\times Y}}$, while the germ $\mathrm{Def}(X,B_Y)\times\mathrm{Def}(Y,B_X)$ is a hull of the functor $\mathrm{Def}_{K_X\otimes B_Y}\times\mathrm{Def}_{K_Y\otimes B_X}$.
By Lemma 4.2 there exists a quasiisomorphism $(B_X\otimes K_Y)\oplus(B_Y\otimes K_X)\to K_{X\times Y}$ and we can apply Theorem 1.1. □

Corollary 4.4. *Let $\mathbb{C}^q/\Gamma$ be a complex torus of dimension q and let Y be a compact Kähler manifold such that $H^1(\mathcal{O}_Y)=H^1(T_Y)=0$. Then*

$$\mathrm{Def}(\mathbb{C}^q/\Gamma\times Y)\simeq\mathbb{C}^{q^2}\times C(q,H^0(T_Y)).$$

Moreover, if $q>1$ then the following three conditions are equivalent:

1. $\mathrm{Def}(\mathbb{C}^q/\Gamma\times Y)$ *is smooth.*
2. *There exists the universal deformations of $\mathbb{C}^q/\Gamma\times Y$.*
3. *The Lie algebra $H^0(Y,T_Y)$ is abelian.*

Proof. The first part and the equivalence [$1 \Leftrightarrow 3$] are immediate consequences of Theorem 4.3. We only prove the equivalence of conditions 2 and 3.
Let $z_1, \ldots, z_q$ be linear coordinates on $\mathbb{C}^q$. Denoting by $\overline{\mathbb{C}^q}^\vee = \langle d\overline{z_1}, \ldots, d\overline{z_q}\rangle$ the space of invariant $(0,1)$-form on the torus, there exists an isomorphism $\bigwedge^* \overline{\mathbb{C}^q}^\vee = B_{\mathbb{C}^q/\Gamma}$. Similarly the inclusion

$$\bigwedge^* \overline{\mathbb{C}^q}^\vee \otimes H^0(T_{\mathbb{C}^q/\Gamma}) \hookrightarrow K_{\mathbb{C}^q/\Gamma}$$

is a quasiisomorphism of differential graded Lie algebras and then the inclusion

$$Q := \bigwedge^* \overline{\mathbb{C}^q}^\vee \otimes \left(H^0(T_{\mathbb{C}^q/\Gamma}) \oplus H^0(T_Y)\right) \hookrightarrow K_{\mathbb{C}^q/\Gamma\times Y}$$

is bijective in H^0, H^1 and injective in H^2.
In particular the functor Def_Q is isomorphic to the functors of deformations of $\mathbb{C}^q/\Gamma \times Y$. Since Q has trivial differential, the functor Def_Q is prorepresentable if and only if the gauge action is trivial or, equivalently, if and only if $[Q^0, Q^1] = 0$. It is straightforward to check that $[Q^0, Q^1] = 0$ if and only if $H^0(T_Y)$ is abelian. □

5. Deformations of $\mathbb{C}^q/\Gamma \times \mathbb{P}^n$

Let $\mathbb{C}^q/\Gamma$ be a complex torus of dimension q and $z_1, \ldots, z_q$ linear coordinates on $\mathbb{C}^q$. For later use and notational purposes, we reprove the first part of Corollary 4.4 when Y is the projective space $\mathbb{P}^n$. In this case, according to Künneth formula, the inclusion

$$Q := \bigwedge^* \overline{\mathbb{C}^q}^\vee \otimes (H^0(T_{\mathbb{C}^q/\Gamma}) \oplus H^0(T_{\mathbb{P}^n})) \hookrightarrow K_{\mathbb{C}^q/\Gamma\times\mathbb{P}^n} \tag{1}$$

is a quasiisomorphism of differential graded Lie algebras.

Corollary 5.1. *The deformation space* $\mathrm{Def}(\mathbb{C}^q/\Gamma \times \mathbb{P}^n)$ *is analytically isomorphic to the germ at 0 of* $\mathbb{C}^{q^2} \times C(q, \mathfrak{sl}(n+1))$. *In particular it is singular for every* $n \geq 1$, $q \geq 2$ *and it is reducible for every* $n \geq 3$ *and* $q \geq 3 + 8/(n-2)$.

Proof. Observe that $H^0(T_{\mathbb{P}^n}) \simeq \mathfrak{sl}(n+1)$ and apply the previous results. □

Our next goal is to prove a similar result for deformations of polarized varieties. More precisely we assume that $\mathbb{C}^q/\Gamma$ is an abelian variety, we fix an ample line bundle L on $\mathbb{C}^q/\Gamma \times \mathbb{P}^n$ and we consider deformations of the pair $(\mathbb{C}^q/\Gamma \times \mathbb{P}^n, L)$.

We recall that an Appell-Humbert (A.-H.) data on a complex torus $\mathbb{C}^q/\Gamma$ is a pair (α, H), where H is a hermitian form on $\mathbb{C}^n$ such that its imaginary part E is integral on $\Gamma \times \Gamma$ and

$$\alpha\colon \Gamma \to \mathrm{U}(1), \qquad \alpha(\gamma_1 + \gamma_2) = \alpha(\gamma_1)\alpha(\gamma_2)(-1)^{E(\gamma_1,\gamma_2)}.$$

Denote by $L(\alpha, H)$ the line bundle on $\mathbb{C}^q/\Gamma$ with factor of automorphy [21, pag. 4]

$$A_\gamma(z) = \alpha(\gamma)e^{\pi(H(z,\gamma)+H(\gamma,\gamma)/2)}, \qquad \gamma \in \Gamma,\ z \in \mathbb{C}^q.$$

It is well known that every line bundle on $\mathbb{C}^q/\Gamma$ is isomorphic to $L(\alpha, H)$ for a unique Appell-Humbert data (α, H); moreover the first Chern class of $L(\alpha, H)$ is equal to the invariant $(1,1)$ form corresponding to E [21, Lemma 3.5]. In particular two line bundles $L(\alpha_1, H_1)$, $L(\alpha_2, H_2)$ have the same Chern class if and only if $H_1 = H_2$.

The same proof of the Appell-Humbert theorem given in [21], with minor and straightforward modifications, shows that every line bundle on $\mathbb{C}^q/\Gamma \times \mathbb{P}^n$ is isomorphic to

$$L(\alpha, H, d) := L(\alpha, H) \boxtimes \mathcal{O}(d)$$

for some A.-H. data (α, H) and some integer d.

Denote for simplicity $X = \mathbb{C}^q/\Gamma \times \mathbb{P}^n$. The Atiyah extension of a line bundle $L(\alpha, H, d)$ is the short exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{D}(L(\alpha, H, d)) \xrightarrow{\sigma} T_X \longrightarrow 0,$$

where $\mathcal{D}(L(\alpha, H, d))$ is the sheaf of first order differential operator on $L(\alpha, H, d)$ and σ is the principal symbol. The induced morphism in cohomology

$$H^1(T_X) = H^1(T_{\mathbb{C}^q/\Gamma}) \otimes H^0(\mathcal{O}_{\mathbb{P}^n}) \oplus H^1(\mathcal{O}_{\mathbb{C}^q/\Gamma}) \otimes H^0(T_{\mathbb{P}^n}) \xrightarrow{\lrcorner c_1} H^2(\mathcal{O}_X)$$

is equal to the contraction with the first Chern class of the line bundle $L(\alpha, H, d)$ ([2]).

Lemma 5.2. *Let c_1 be the first Chern class of the line bundle $L(\alpha, H, d)$ on $X = \mathbb{C}^q/\Gamma \times \mathbb{P}^n$ and assume that* $\det(H) \neq 0$. *Then $\lrcorner c_1(H^1(\mathcal{O}_{\mathbb{C}^q/\Gamma}) \otimes H^0(T_{\mathbb{P}^n})) = 0$ and*

$$H^1(T_{\mathbb{C}^q/\Gamma}) \otimes H^0(\mathcal{O}_{\mathbb{P}^n}) \xrightarrow{\lrcorner c_1} H^2(\mathcal{O}_X)$$

is surjective; in particular $H^2(\mathcal{D}(L(\alpha, H, d))) \longrightarrow H^2(T_X)$ is injective.

Proof. The first part is clear since $\lrcorner c_1(H^1(\mathcal{O}_{\mathbb{C}^q/\Gamma}) \otimes H^0(T_{\mathbb{P}^n})) \subset H^1(\mathcal{O}_{\mathbb{C}^q/\Gamma}) \otimes H^1(\mathcal{O}_{\mathbb{P}^n})$. The map

$$H^1(T_{\mathbb{C}^q/\Gamma}) \otimes H^0(\mathcal{O}_{\mathbb{P}^n}) \xrightarrow{\lrcorner c_1} H^2(\mathcal{O}_X) = H^2(\mathcal{O}_{\mathbb{C}^q/\Gamma}) \otimes H^0(\mathcal{O}_{\mathbb{P}^n})$$

can be written as $\lrcorner c_1 = e \otimes Id$, where Id is the identity on $H^0(\mathcal{O}_{\mathbb{P}^n})$ and

$$e\colon H^1(T_{\mathbb{C}^q/\Gamma}) \to H^2(\mathcal{O}_{\mathbb{C}^q/\Gamma})$$

is the contraction with the first Chern class of $L(\alpha, H)$.
The elements of $H^2(\mathcal{O}_{\mathbb{C}^q/\Gamma})$ are represented by invariant $(0,2)$-forms; more precisely, if $z_1, \ldots, z_q$ are linear coordinates on $\mathbb{C}^q$, then a basis of $H^2(\mathcal{O}_{\mathbb{C}^q/\Gamma})$ is given by the forms $d\overline{z}_i \wedge d\overline{z}_j$, for $i < j$. Similarly a basis of $H^1(T_{\mathbb{C}^q/\Gamma})$ is given by the invariant tensors $d\overline{z}_i \otimes \dfrac{\partial}{\partial z_j}$, for $i, j = 1, \ldots, q$.
The first Chern class of $L(\alpha, H)$ is given by the invariant form $\sum h_{rs} dz_r \wedge d\overline{z}_s$, ,with (h_{rs}) a scalar multiple of H, and

$$e\left(d\overline{z}_i \otimes \frac{\partial}{\partial z_j}\right) = d\overline{z}_i \otimes \frac{\partial}{\partial z_j} \lrcorner \sum_{r,s} h_{rs} dz_r \wedge d\overline{z}_s = \sum_s h_{js} d\overline{z}_i \wedge d\overline{z}_s.$$

Therefore the surjectivity of the matrix (h_{rs}) implies the surjectivity of e. $\square$

Theorem 5.3. *Let L be a line bundle on $X = \mathbb{C}^q/\Gamma \times \mathbb{P}^n$ and denote by* $\mathrm{Def}(X, L)$ *the deformation space of the pair (X, L).*

1. *If $L \in$* $\mathrm{Pic}^0(X)$, *then the natural morphism* $\mathrm{Def}(X, L) \to \mathrm{Def}(X)$ *is smooth of relative dimension q.*
2. *If L is ample then there exists a smooth morphism of analytic germs*

$$\mathrm{Def}(X, L) \to C(q, \mathfrak{sl}(n+1, \mathbb{C})).$$

Proof. The first part is a general result which is true for every Kähler manifold X. In fact by Deligne's theorem [6], for every deformation $\mathcal{X}\to Spec(A)$ of X, with A a local Noetherian ring, the group $H^1(\mathcal{O}_{\mathcal{X}})$ is a free A-module of rank $h^1(\mathcal{O}_X)$ and then there are no obstructions to extend a topologically trivial line bundles over any deformation of X.
Assume now that L is ample; then $L=L(\alpha,H,d)$ for some $d>0$ and H positive definite. Consider the Dolbeault resolution of the Atiyah extension of L

$$0\longrightarrow A_X^{0,*}\longrightarrow A_X^{0,*}(\mathcal{D}(L))\xrightarrow{\sigma} A_X^{0,*}(T_X)\longrightarrow 0.$$

The differential graded Lie algebra $A_X^{0,*}(\mathcal{D}(L))$ governs the deformations of the pair (X,L) (see [5]).
Denoting by P the fiber product of σ and the injective quasiisomorphism $Q\to A_X^{0,*}(T_X)$ of Equation (1), we have a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A_X^{0,*} & \longrightarrow & P & \longrightarrow & Q & \longrightarrow & 0\\
 & & \| & & \downarrow\phi & & \downarrow & & \\
0 & \longrightarrow & A_X^{0,*} & \longrightarrow & A_X^{0,*}(\mathcal{D}(L)) & \xrightarrow{\sigma} & A_X^{0,*}(T_X) & \longrightarrow & 0
\end{array}$$

with exact rows and vertical quasiisomorphisms. In particular, according to Theorem 1.1, the functor Def_P is isomorphic to the functor of deformations of the pair (X,L).
The subspace $I=\bigwedge^1\overline{\mathbb{C}^q}^\vee\otimes H^0(T_{\mathbb{C}^q/\Gamma})\subset Q^1$ is an ideal of the DGLA Q; denote by $R=Q/I$ the corresponding quotient DGLA. Since $R^1=\bigwedge^1\overline{\mathbb{C}^q}^\vee\otimes H^0(T_{\mathbb{P}^n})$, the commuting variety $C(q,\mathfrak{sl}(n+1,\mathbb{C}))$ is exactly the deformation space of the functor Def_R. Therefore it is sufficient to prove that the morphism $P\to R$ is surjective on H^1 and injective on H^2; this follows easily from Lemma 5.2. □

6. Natural deformations of complete intersections

Let X be a smooth complex manifold of dimension n and let $D_1,\dots,D_m$ be smooth divisors with $0<m\le n-2$. Assume that $D_1,\dots,D_m$ intersect transversally on a smooth subvariety S of codimension m.
The natural deformations of S are the deformations obtained by deforming X and the divisors D_i. More precisely, let $\mathrm{Def}_{X;D_1,\dots,D_m}\colon \mathbf{Art}\to\mathbf{Set}$ be the functor of infinitesimal deformations of the holomorphic map $\overset{\circ}{\cup} D_i\to X$. Equivalently every element of $\mathrm{Def}_{X;D_1,\dots,D_m}(A)$ is the data of a deformation $\mathcal{X}\to Spec(A)$ of X and deformations $\mathcal{D}_i\subset\mathcal{X}$ of the smooth hypersurfaces D_i. Since $\cap_i\mathcal{D}_i$ is a deformation of S over $Spec(A)$, it is well defined a natural transformation

$$\mathrm{Nat}\colon\ \mathrm{Def}_{X;D_1,\dots,D_m}\to\mathrm{Def}_S\,.$$

The (infinitesimal) natural deformations of S are the ones lying in the image of Nat.

The aim of this section is to give a sufficient condition for the completeness of the natural deformations of S. Since S is complete intersection, the ideal sheaf $\mathcal{I}_S\subset\mathcal{O}_X$ admits the Koszul resolution

$$0\longrightarrow\mathcal{O}_X(-\sum_{i=1}^m D_i)\longrightarrow\cdots\longrightarrow\mathop{\oplus}_{i<j}\mathcal{O}_X(-D_i-D_j)\longrightarrow\mathop{\oplus}_i\mathcal{O}_X(-D_i)\longrightarrow\mathcal{I}_S\longrightarrow 0.$$

Denote by $T_X(-\log D) \subset T_X$ the subsheaf of vector fields which are tangent to D_i for every i, and by $N_{D_i|X} \simeq \mathcal{O}_{D_i}(D_i)$ the normal bundle of D_i in X. There exists a short exact double sequence:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{L} & \longrightarrow & T_X \otimes \mathcal{I}_S & \longrightarrow & \oplus_i \mathcal{I}_S N_{D_i|X} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & T_X(-\log D) & \longrightarrow & T_X & \longrightarrow & \oplus_i N_{D_i|X} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\alpha} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & T_S & \longrightarrow & T_X \otimes \mathcal{O}_S & \longrightarrow & N_{S|X} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Theorem 6.1. *For every $A \subset \{1, \dots, m\}$ denote by $D_A = \sum_{i \in A} D_i$. Assume that*

1. *$H^{|A|+1}(T_X(-D_A)) = 0$ for every $A \neq \emptyset$.*
2. *$H^{|A|}(\mathcal{O}_X(D_i - D_A)) = 0$ for every $i = 1, \dots, m$ and every $A \neq \{i\}, \emptyset$.*

Then the natural transformation $\mathrm{Nat}\colon \mathrm{Def}_{X;D_1,\dots,D_m} \to \mathrm{Def}_S$ *is smooth.*

Proof. Considering the cohomology of the tensor product of T_X with the Koszul resolution of $\mathcal{I}_S$, we get immediately that the first condition implies that $H^2(T_X \otimes \mathcal{I}_S) = 0$. Let i be fixed; assume that $A \neq \emptyset$ and $i \notin A$, then from the exact sequence

$$0 \longrightarrow \mathcal{O}_X(D_i - D_{A \cup \{i\}}) \longrightarrow \mathcal{O}_X(D_i - D_A) \longrightarrow \mathcal{O}_{D_i}(D_i - D_A) \longrightarrow 0$$

we get $H^{|A|}(\mathcal{O}_{D_i}(D_i - D_A)) = 0$. Considering the cohomology of the tensor product of $\mathcal{O}_{D_i}(D_i)$ with the Koszul resolution of the ideal of S in D_i we get that $H^1(\mathcal{I}_S N_{D_i|X}) = 0$. Therefore $H^2(\mathcal{L}) = 0$ and the morphism $\alpha\colon T_X(-\log D) \to T_S$ is surjective in H^1 and injective in H^2.
By general results of deformation theory (see e.g. [31]) tangent and obstruction spaces of the functor $\mathrm{Def}_{X;D_1,\dots,D_m}$ are $H^1(T_X(-\log D))$ and $H^2(T_X(-\log D))$ respectively. Therefore Nat is surjective on tangent spaces, injective on obstruction spaces and therefore it is smooth. □

Remark 6.2. For $m = 1$ the Theorem 6.1 reduces to a particular case of Horikawa's costability theorem.

7. Some new example of obstructed irregular surfaces

In order to apply Theorem 6.1 to the variety $X = \mathbb{C}^q/\Gamma \times \mathbb{P}^n$ we need the determination of cohomology groups of the line bundles $L(\alpha, H, d)$. Notice that $K_X = L(0, 0, -n-1)$ and

$$L(\alpha_1, H_1, d_1) \otimes L(\alpha_2, H_2, d_2) = L(\alpha_1\alpha_2, H_1 + H_2, d_1 + d_2).$$

Lemma 7.1. *Let $X = \mathbb{C}^q/\Gamma \times \mathbb{P}^n$, then*

1. *If $\alpha \neq 1$ then $H^i(L(\alpha, 0, d)) = 0$ for every $i, d \in \mathbb{Z}$.*
2. *If H is positive definite and $d \geq -n$, then $H^i(L(\alpha, H, d)) = 0$ for every $i > 0$.*

(3) *If H is negative definite and $d \le -2$, then $H^i(L(\alpha,H,d)) = H^i(T_X \otimes L(\alpha,H,d)) = 0$ for every $i \le q+n-2$.*
(4) *If H is negative definite and $d \le -n-2$, then $H^{q+n-1}(T_X \otimes L(\alpha,H,d)) = 0$.*

Proof. The determination of the cohomology of line bundles on $\mathbb{C}^q/\Gamma$ [21, Th. 3.9] gives that:

- If $\alpha \neq 1$ then $H^i(L(\alpha,0)) = 0$ for every i,.
- If H is negative definite, then $H^i(L(\alpha,H)) = 0$ for every $i < q$.

By Künneth formula we get item (1). Assume now that H is negative definite; then by Künneth formula we get $H^i(L(\alpha,H,d)) = 0$ for every α, every $d < 0$ and every $i < q+n$; Serre duality implies (2).
The bundle $T_X \otimes L(\alpha,H,d)$ is the direct sum of $L(\alpha,H) \boxtimes T_{\mathbb{P}^n}(d)$ and q copies of $L(\alpha,H,d)$. Since $H^i(T_{\mathbb{P}^n}(d)) = 0$ for every $d \le -2$ and every $i \le n-2$ we get item (3). If $d \le -n-2$ then $H^{n-1}(T_{\mathbb{P}^n}(d)) = 0$ and this implies item (4). $\square$

Definition 7.2. Let D be a divisor of a section of the line bundle $L(\alpha,H,d)$. We shall call the pair (H,d) the *homology type* of D.

The above definition is justified since the Poincaré dual of D is the sum of d times the hyperplane class on $\mathbb{P}^n$ and the class represented by the imaginary part of H.

Proposition 7.3. *Assume $q,n,d \ge 2$ and $q+n+d \ge 7$. Let S be a smooth ample hypersurface of $\mathbb{C}^q/\Gamma \times \mathbb{P}^{n-1}$ of homology type (H,d). Then every deformation of S is projective and* $\mathrm{Def}(S)$ *has the same singularity type of the commuting variety $C(q,\mathfrak{sl}(n,\mathbb{C}))$.*

Proof. The hypersurface S is the zero locus of a section s of a line bundle $L = L(\alpha,H,d)$ for some semi-character α. Since S is ample the hermitian form H is positive definite and then, according to Theorem 5.3 there exists a smooth morphism

$$\mathrm{Def}(X,L) \to C(q,\mathfrak{sl}(n,\mathbb{C})).$$

Since $H^1(L) = 0$, the section s extends to every deformation of the pair (X,L) and then the natural morphism $\mathrm{Def}(X,S) \to \mathrm{Def}(X,L)$ is smooth. The ampleness of S also implies that every deformation of the pair (X,S) is projective.
On the other hand, by Lemma 7.1 $H^2(T_X(-S)) = 0$ and then by Theorem 6.1 the morphism $\mathrm{Def}(X,S) \to \mathrm{Def}(S)$ is smooth. $\square$

Remark 7.4. The fact that obstructions to smoothness of $\mathrm{Def}(X,S) \to \mathrm{Def}(X,L)$ are contained in $H^1(L)$ is well known and easy to prove directly. However, it is interesting to see this fact also as a consequence of the exact sequence

$$0 \to T_X(-\log S) \xrightarrow{\alpha} \mathcal{D}(L) \xrightarrow{\beta} L \to 0,$$

$$\alpha(\chi) = \chi - s^{-1}\chi(s), \qquad \beta(\phi) = \phi(s).$$

Notice that α is a morphism of sheaves of Lie algebras.

Example 7.5. Let A be an abelian surface and S a smooth surface of general type contained in $A \times \mathbb{P}^1$. Then $\mathcal{O}(S) = L(\alpha,H,d)$ for some H positive definite and $d \ge 3$; according to Proposition 7.3 $\mathrm{Def}(S)$ has the singularity type of $C(2,\mathfrak{sl}(2,\mathbb{C}))$. Since two matrices in $\mathfrak{sl}(2,\mathbb{C})$ commute if and only if they are linearly dependent, the commuting variety $C(2,\mathfrak{sl}(2,\mathbb{C}))$ is equal to determinantal variety of matrices 2×3 of rank ≤ 1.

Theorem 7.6. *Let S be a smooth complete intersection of $m = q+n-3$ smooth divisors $D_1,\ldots,D_m$ of $X = \mathbb{C}^q/\Gamma \times \mathbb{P}^{n-1}$. Assume that $\mathcal{O}(D_i) = L(\alpha_i, H, d)$ with H is positive definite, $d \geq 2$, $d(q+n-3) \geq n+1$ and $\alpha_i \neq \alpha_j$ for every $i \neq j$.*
Then every deformation of S is projective and $\operatorname{Def}(S)$ has the same singularity type of the commuting variety $C(q, \mathfrak{sl}(n,\mathbb{C}))$.
In particular, if $q, n \geq 2$ then S is obstructed and $\operatorname{Def}(S)$ is reducible for $n \geq 4$ and $q \geq 3 + 8/(n-3)$.

Proof. By adjunction formula the surface S has ample canonical bundle and then every deformation of S is projective.
By Lemma 7.1 and Theorem 6.1 the morphism $\mathrm{Nat}\colon \operatorname{Def}(X; D_1,\ldots,D_m) \to \operatorname{Def}(S)$ is smooth. According to Theorem 5.3 there exists a smooth morphism

$$\operatorname{Def}(X, L(\alpha_1, H, d)) \to C(q, \mathfrak{sl}(n,\mathbb{C})).$$

Therefore it is sufficient to prove that that the morphism

$$\operatorname{Def}(X; D_1,\ldots,D_m) \to \operatorname{Def}(X; \mathcal{O}(D_1))$$

is smooth. First of all, since $H^1(D_i) = 0$ for every i the morphism

$$\operatorname{Def}(X; D_1,\ldots,D_m) \to \operatorname{Def}(X; \mathcal{O}(D_1),\ldots,\mathcal{O}(D_m))$$

is smooth. On the other side, since $\mathcal{O}(D_i - D_j) \in \operatorname{Pic}^0(X)$, by Theorem 5.3 also the morphism

$$\operatorname{Def}(X; \mathcal{O}(D_1),\ldots,\mathcal{O}(D_m)) \to \operatorname{Def}(X; \mathcal{O}(D_1))$$

is smooth. □

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